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Stability of asymptotically stable sets for continuous functions

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Abstract

Let K be the class of nonempty compact subsets of $I = [0,1]$, and K^* consist of the nonempty closed subsets of K. We study the map $\overline{A}: C(I,I) \to K^*$ defined so that $\overline{A(f)}$ is the closure of the collection of asymptotically stable sets of f. While we find that $\overline{A}$ is not continuous, we do get much more positive results when studying the semicontinuity of $\overline{A}$. In particular, $\overline{A}$ is lower semicontinuous at a residual set of functions in $C(I,I)$, so that $\overline{A}$ is continuous if we restrict our map to a residual subset of $C(I,I)$.

Keywords: Asymptotically stable set; Attractor; ω-limit set; Semicontinuity

1. Introduction

At the Twentieth Summer Symposium in Real Analysis, A.M. Bruckner posed several questions regarding the iterative stability of continuous functions as they experience small perturbations, and discussed why these questions are of general interest [3]. In particular, how are the set of ω-limit points and the collection of

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ω-limit sets affected by slight changes in the generating function? As one sees from various examples found in [3] and [8], in general, both the set of ω-limit points and the collection of ω-limit sets of a typical continuous function are affected dramatically by arbitrarily small perturbations. We found in [9], however, that by restricting ourselves to certain classes of functions, one gets more positive results.

Theorem 1.1. *The map* $\Lambda : (C(I, I), \| \circ \|) \to (K, H)$ *given by*

$$f \to \Lambda(f) = \bigcup_{x\in I} \omega(x, f)$$

is continuous at f *if and only if* $\overline{S(f)} = CR(f)$; *that is, the closure of the set of stable periodic points of* f *is the set of its chain recurrent points.*

Theorem 1.2. *Let* $E = \{f \in C(I, I)\colon\ f$ *possesses zero topological entropy*$\}$. *Then* $\Omega : (E, \| \circ \|) \to (K^*, H^*)$ *given by*

$$f \to \Omega(f) = \{\omega(x, f)\colon\ x \in I\}$$

is continuous at f *if and only if either of the following equivalent conditions hold*:

(1) $\overline{S(f)} = CR(f)$.
(2) *Every periodic orbit of* f *is stable, and every simple system of* f *has non-empty interior.*

In this paper we continue our study of the stability of iterative systems by considering asymptotically stable sets. After presenting the notation, definitions and previously known results we will use throughout the balance of the paper in Section 2, we begin our study of asymptotically stable sets in Section 3. There we find that $A(f) = \{\alpha\colon\ \alpha$ is an asymptotically stable set of $f \in C(I, I)\}$ is not always closed with respect to the Hausdorff metric H, in marked contrast to $\Omega(f)$, which is always closed [2]. In Section 4 we investigate the semicontinuity of the map $f \to \overline{A(f)}$. While this map is, in general, neither upper nor lower semicontinuous, we find that quite a bit can be said about the persistence of asymptotically stable sets using the notion of semicontinuity. Should we consider the effect that perturbing f has on one of its particular asymptotically stable sets α and its associated basin of attraction $B(\alpha)$, we find that α will always shrink (or at least not get much larger), while $B(\alpha)$ will always grow (or at least not get much smaller). We also find that the map $f \to \overline{A(f)}$ is lower semicontinuous whenever $\Lambda : (C(I, I), \| \circ \|) \to (K, H)$ given by $f \to \Lambda(f) = \bigcup_{x\in I} \omega(x, f)$ is lower semicontinuous. In Section 5 we develop a residual subset T of $C(I, I)$ such that every element of T possesses a nontrivial asymptotically stable set, $f \to \overline{A(f)}$ is lower semicontinuous at every element of T, and the map $f \to \overline{A(f)}$, when restricted to elements of T, is continuous.

2. Preliminaries

We shall be concerned with the class $C(I,I)$ of continuous functions mapping the unit interval $I=[0,1]$ into itself, and the iterative properties this class of functions possesses. For f in $C(I,I)$ and any integer $n \geqslant 1$, f^n denotes the nth iterate of f. Let $P(f)$ represent those points $x \in I$ that are periodic under f, and if x is a periodic point of period n for which $f^n(x)-x$ is not unisigned in any deleted neighborhood of x, then x is called a *stable periodic point*; we let $S(f)$ represent the stable periodic points of f, and let $P_n(x)=\{x \in I\colon\ f^n(x)=x,\ f^m(x) \neq x \text{ whenever } m \mid n\}$ represent the f-periodic points with period n. For each x in I, we call the set of all subsequential limits of the sequence $\{f^n(x)\}_{n=0}^{\infty}$ the *ω-limit set of f generated by x*, and write $\omega(x,f)$. Let $\Lambda(f)=\bigcup_{x\in I}\omega(x,f)$ represent the ω-limit points of f, while $\Omega(f)=\{\omega(x,f)\colon\ x \in I\}$ denotes the set composed of the ω-limit sets of f. Now, let $\varepsilon>0$ be given, and take x and y to be any points in $[0,1]$. An *ε-chain* from x to y with respect to a function f is a finite set of points $\{x_0,x_1,\ldots,x_n\}$ in $[0,1]$ with $x=x_0$, $y=x_n$ and $|f(x_{k-1})-x_k|<\varepsilon$ for $k=1,2,\ldots,n$. We call x a *chain recurrent point* of f if there is an ε-chain from x to itself for any $\varepsilon>0$, and write $x \in CR(f)$. We note that for every f in $C(I,I)$, $\Lambda(f) \subseteq CR(f)$.

In addition to the usual, Euclidean metric d on $I=[0,1]$, we will be working in three metric spaces. Within $C(I,I)$ we will use the supremum metric given by $\|f-g\|=\sup\{|f(x)-g(x)|\colon\ x\in I\}$. Our second metric space (K,H) is composed of the class of nonempty closed sets K in I endowed with the Hausdorff metric H given by $H(E,F)=\inf\{\delta>0\colon\ E \subset B_\delta(F),\ F\subset B_\delta(E)\}$, where $B_\delta(F)=\{x\in I\colon\ d(x,y)<\delta,\ y\in F\}$. This space is compact [4]. Our final metric space (K^*,H^*) consists of the nonempty closed subsets of K. Thus, $K \in K^*$ if K is a nonempty family of nonempty closed sets in I such that K is closed in K with respect to H. We endow K^* with the metric H^* so that K_1 and K_2 are close with respect to H^* if each member of K_1 is close to some member of K_2 with respect to H, and vice versa. This metric space is also compact [10]. Our interest in, and the utility of, the spaces (K,H) and (K^*,H^*) stem from the following two theorems from [1] and [2], respectively.

Theorem 2.1. *For any f in $C(I,I)$, the set $\Lambda(f)$ is closed in I.*

Theorem 2.2. *For any f in $C(I,I)$, the set $\Omega(f)$ is closed in (K,H).*

In this paper we study the stability of iterative structures by considering asymptotically stable sets. If f is an element of $C(I,I)$, then a nonempty closed set F contained in I is *Lyapunov stable* if, for each open set U containing F, there exists an open set V containing F such that $f^n(x)\in U$ for all $x\in V$, for all natural numbers n. A nonempty closed set F is an *attractor* if there exists an open set B containing F such that $\omega(x,f)\subset F$ for every $x\in B$. If a nonempty closed set α is both Lyapunov stable and an attractor, we say that α is *asymptotically stable*. Furthermore, an asymptotically stable set is indecomposable if it cannot be represented as the union of two disjoint closed invariant proper subsets. We let $A(f)$ represent a continuous function's collection of asymptotically stable sets, so that $A(f) = \{\alpha\colon \alpha \text{ is an asymptotically stable set of } f\}$. In the ensuing sections we make use of the following results found in [1]; these also shed some light onto the properties of asymptotically stable sets.

Proposition 2.3. *If α is a closed invariant set and if there exists an open set $V \supseteq \alpha$ such that*

(1) $f(V) \subset V$,
(2) $\bigcap_{n\geqslant 0} f^n(\overline{V}) \subseteq \alpha$,

then α is an asymptotically stable set.

Corollary 2.4. *If U is a nonempty open set such that $f(\overline{U}) \subseteq U$, then $\alpha = \bigcap_{n\geqslant 0} f^n(\overline{U})$ is asymptotically stable.*

Let us now turn our attention to the Baire category theorem. Let (X, ρ) be a metric space. A set is of the first category in (X, ρ) if it can be written as a countable union of nowhere dense sets; otherwise, the set is of the second category. A set is residual if it is the complement of a first category set; an element of a residual subset of (X, ρ) is called a typical element of X. With these definitions in mind, we recall Baire's theorem on category.

Theorem 2.5. *Let (X, ρ) be a complete metric space, with S a first category subset of X. Then $X - S$ is dense in X.*

Since much of our work will take place in the complete metric space $(C(I, I), \|\circ\|)$, where $\|f - g\| = \sup\{|f(x) - g(x)|\colon x \in I\}$, we will have the opportunity to make good use of Baire's theorem.

3. The algebraic structure of $A(f)$

As is pointed out in [1], finite unions of asymptotically stable sets are also asymptotically stable, as is their intersection provided it is nonempty; these conclusions follow readily from the definition of an asymptotically stable set. We now turn our attention to countable unions of asymptotically stable sets.

Example 3.1. There exist functions $f \in C(I, I)$ with asymptotically stable sets $\{\alpha_n\}_{n=1}^{\infty}$ so that neither $\bigcup_{n=1}^{\infty} \alpha_n$ nor $\overline{\bigcup_{n=1}^{\infty} \alpha_n}$ is asymptotically stable.

Construction. Let $\alpha_n = 1/2 + 1/2^{n+1}$ and set

$$B_n = \left(\frac{1}{2} + \frac{1}{2^{n+1}} - \frac{1}{2^{n+3}}, \frac{1}{2} + \frac{1}{2^{n+1}} + \frac{1}{2^{n+3}}\right).$$

We construct f so that

(1) $f(1/2 + 1/2^{n+1}) = 1/2 + 1/2^{n+1}$ for $n = 1, 2, 3, \ldots,$
(2) f is a contraction map on B_n for $n = 1, 2, 3, \ldots,$
(3) f is linear on $[1/4, 1/2]$ with $f(1/2) = 1/2$ and $f(1/4) = 1$,
(4) $f(x) = 1$ for some

$$x \in \left(\frac{1}{2} + \frac{1}{2^{n+2}} + \frac{1}{2^{n+4}}, \frac{1}{2} + \frac{1}{2^{n+1}} - \frac{1}{2^{n+3}}\right),$$

if

$$\left(\frac{1}{2} + \frac{1}{2^{n+1}} - \frac{1}{2^{n+3}}\right) > \frac{5}{8},$$

(5) $f(x) = -4x + 5/2$ for some

$$x \in \left(\frac{1}{2} + \frac{1}{2^{n+2}} + \frac{1}{2^{n+4}}, \frac{1}{2} + \frac{1}{2^{n+1}} - \frac{1}{2^{n+3}}\right),$$

if

$$\left(\frac{1}{2} + \frac{1}{2^{n+1}} - \frac{1}{2^{n+3}}\right) < \frac{5}{8},$$

and
(6) f is continuous on $[0, 1]$.

It follows, then, that α_n is an asymptotically stable set for f, for each n, as is $\bigcup_{n=1}^{k} \alpha_n$ [1]. By our construction of f and the α_n, however, neither $\overline{\bigcup_{n=1}^{\infty} \alpha_n}$ nor $\overline{\bigcup_{n=1}^{\infty} \alpha_n}$ is asymptotically stable, as $\bigcup_{n=1}^{\infty} \alpha_n$ is not closed, and $\overline{\bigcup_{n=1}^{\infty} \alpha_n}$ is not Lyapunov stable.

Not only does our example allow us to assert that the closure of countable unions of asymptotically stable sets may not themselves be asymptotically stable, but also to develop the following proposition.

Proposition 3.2. *There exist functions $f \in C(I, I)$ for which $A(f)$ is not closed with respect to the Hausdorff metric.*

Proof. Referring to our construction in Example 3.1, we see that $\{1/2\} = \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} 1/2 + 1/2^{n+1}$ must be an asymptotically stable set of f should $A(f)$ be closed. That $\{1/2\}$ is not an asymptotically stable set of f follows readily from our construction. $\square$

Our proposition makes evident a considerable cleavage between the sets $A(f)$ and $\Omega(f) = \{\omega(x, f)\colon\ x \in I\}$, as the latter is always closed with respect to the Hausdorff metric whenever f is a continuous function [2]. Our example also shows that if $\{\alpha_n\}_{n=1}^{\infty}$ is a nondecreasing sequence of asymptotically stable sets, it does not necessarily follow that the closure of their union, or limit in (K, H), is also asymptotically stable. Again, this is in contrast to a nondecreasing sequence of ω-limit sets, the closure of their union always being an ω-limit set. In fact, the maximal ω-limit sets that may result from such nondecreasing sequences have received considerable attention, and in some ways well represent the chaotic nature of the generating function [5,7].

We now turn our attention to the properties of those continuous functions for which $A(f)$ is closed.

Proposition 3.3. *Let $f \in C(I, I)$ for which $A(f)$ is closed in (K, H) with $\{\alpha_n\}_{n=1}^{\infty} \subset A(f)$ such that* $\lim_{n\to\infty} \alpha_n = \alpha$. *Then there exists N a natural number such that*

(1) *$\alpha_n \cap \Lambda(f) \subseteq \alpha$ for all $n > N$, and*
(2) *if $\omega \in \Omega(f)$ with $\omega \subset \alpha_n$ for some $n > N$, then $\omega \subset \alpha$.*

Proof. Since $A(f)$ is closed in (K, H), $\alpha \in A(f)$. Let U be an open set containing α so that $\omega(x, f) \subseteq \alpha$ for all x in U. Since $\lim_{n\to\infty} \alpha_n = \alpha$, there exists N a natural number so that $\alpha_m \subseteq U$ whenever $m > N$. Fix $n > N$. Since $\alpha_n \in A(f)$, there exists U_n so that $\alpha_n \subseteq U_n \subseteq U$, and $\omega(x, f) \subseteq \alpha_n$ for all x in U_n. From our choice of U, one sees that $\omega(x, f) \subseteq \alpha$, too. Our conclusion now follows from the observation that, for any $y \in \alpha_n \cap \Lambda(f)$, there exists $x \in U_n$ such that $y \in \omega(x, f)$. $\square$

From our proposition we see that, whenever $A(f)$ is closed in (K, H), all sequences of asymptotically stable sets must terminate in a largest asymptotically stable set, largest in the sense that it contains all the ω-limit points as well as the ω-limit sets found in all elements of the sequence, beyond a certain point. We note that the limit of a sequence of asymptotically stable sets must always contain an ω-limit set, even if the limit set itself is not asymptotically stable. This follows from [2].

4. The semicontinuity of the map $f \mapsto \overline{A(f)}$

We begin now our study of $A(f)$, and how this collection of asymptotically stable sets is affected by perturbations in f. As we saw in the previous section, $A(f)$ is not always closed in (K, H) for an arbitrary f in $C(I, I)$. Because of this, we consider the map $\overline{A}: (C(I,I), \|\circ\|) \to (K^*, H^*)$ given by $f \mapsto \overline{A(f)}$, as the closure of $A(f)$ is always contained in (K, H).

We first show that $\overline{A}: (C(I,I), \|\circ\|) \to (K^*, H^*)$ is not upper semicontinuous.

Example 4.1. The map $\overline{A}: (C(I,I), \|\circ\|) \to (K^*, H^*)$ given by $f \mapsto \overline{A(f)}$ is not upper semicontinuous. That is, there exists a sequence $\{f_n\} \subset C(I,I)$ with $\alpha_n \in A(f_n)$ for each n such that $f_n \to f$ in $(C(I,I), \|\circ\|)$ and $\alpha_n \to L$ in (K, H), yet L is not an element of $A(f)$.

Construction. Let $f_n: I \to I$ be the line passing through the points $(0, 1/n)$ and $(1, 1 - 1/n)$ for $n = 2, 3, 4, \ldots$. Then $\{1/2\} \in A(f_n)$ for each n, but $\{1/2\}$ is not an asymptotically stable set for the limit function $f(x) = x$.

We note that, in this example, not only is $\overline{A}: (C(I,I), \|\circ\|) \to (K^*, H^*)$ discontinuous at $f(x) = x$, but so are $\Omega: (C(I,I), \|\circ\|) \to (K^*, H^*)$ given by $f \mapsto \Omega(f)$ and $\Lambda: (C(I,I), \|\circ\|) \to (K, H)$ given by $f \mapsto \Lambda(f)$ [9].

After establishing our next proposition, we are able to develop a notion of lower semicontinuity for $\overline{A}: (C(I,I), \|\circ\|) \to (K^*, H^*)$. Perhaps more importantly, the proposition shows that the asymptotically stable sets of a continuous function display a large degree of stability as the function is perturbed. Specifically, if α is an asymptotically stable set of a continuous function f, and $B(\alpha)$ is its associated basin of attraction, then α shrinks (or at least does not get much larger) and $B(\alpha)$ grows (or at least does not get much smaller) as f experiences small perturbations.

Proposition 4.2. *Let α_f be an asymptotically stable set for f in $C(I,I)$, with $B(\alpha_f)$ its associated asymptotic basin, and $\dot{\varepsilon} > 0$. There exists $\delta > 0$ so that, if $g \in C(I,I)$ for which $\|f - g\| < \delta$, then there is α_g in $A(f)$ such that $\alpha_g \subset B_\varepsilon(\alpha_f)$ and $B(\alpha_f) \subset B_\varepsilon(B(\alpha_g))$.*

Proof. Since α_f is an asymptotically stable set of f, there exists an open set W containing α_f such that $f(\overline{W}) \subset W$, and $\overline{W} \subseteq B_\varepsilon(\alpha_f)$. Let B^* be an open set contained in $B(\alpha_f)$ such that $H(B^*, B(\alpha_f)) < \varepsilon$ and $\overline{B^*} \subset B(\alpha_f)$. By Lemma 3.1.2 of [6], there exists a positive number N with the property that $f^N(\overline{B^*}) \subset W$, so that $f^{N+1}(\overline{B^*}) \subset f(W) \subset f(\overline{W}) \subset W$. Let ∂W denote the boundary of W, with $\gamma = \min\{d(f(x), \partial W)\colon x \in \overline{W}\}$, and choose $\delta > 0$ so that $\|f - g\| < \delta$ implies $\|f^k - g^k\| < \gamma$ for $1 \leqslant k \leqslant N + 1$. It follows, then, that $g^{N+1}(\overline{B^*}) \subset W$ and that $g(\overline{W}) \subset W$. Thus $\alpha_g = \bigcap_{n \geqslant 0} g^n(\overline{W})$ is an asymptotically stable set for g, and by choosing W so that $\overline{W} \subseteq B_\varepsilon(\alpha_f)$, one sees that $\alpha_g \subseteq \overline{W} \subseteq B_\varepsilon(\alpha_f)$. Moreover, if $x \in B^*$, then $\omega(x, g) \subset \alpha_g$, so that $B^* \subset B(\alpha_g)$. Thus, $B(\alpha_f) \subset B_\varepsilon(B(\alpha_g))$. $\square$

Our next example shows that we cannot, in general, sharpen the domination notions found in the conclusion of Proposition 4.2.

Example 4.3. There exists $f \in C(I, I)$ possessing the asymptotically stable set α_f so that the Lebesgue measure $\lambda\alpha_f > 0$ and $\lambda B(\alpha_f) < 1$, but for any $\varepsilon > 0$ there exists g in $C(I, I)$ with a unique $\alpha_g \in A(g)$ so that $\|f - g\| < \varepsilon$, $\alpha_g \subset \alpha_f$, $\lambda\alpha_g = 0$, and $\lambda B(\alpha_g) = 1$.

Construction. We define $f : I \to I$ so that $f(0) = 1/9$, $f(1/9) = 1/9$, $f(2/9) = 2/9 + 2/27$, $f(1/3) = 1/3$, $f(2/3) = 2/3$, $f(7/9) = 2/3 + 1/27$, $f(8/9) = 8/9$, and $f(1) = 8/9$. We now extend f linearly to the remaining complementary intervals. Then $\alpha_f = [1/3, 2/3]$ is an asymptotically stable set of f, with $B(\alpha_f) = (1/9, 8/9)$. Now, let $\varepsilon > 0$ with $0 < \gamma < \min\{\varepsilon, 1/81\}$. Define $g : I \to I$ so that $g(0) = 1/9 - \gamma$, $g(1/9) = 1/9 + \gamma$, $g(2/9) = 2/9 + 2/27 - \gamma$, $g(1/3) = 1/3 + \gamma$, $g(2/3) = 2/3 - \gamma$, $g(7/9) = 2/3 + 1/27 + \gamma$, $g(8/9) = 8/9 - \gamma$ and $g(1) = 8/9 + \gamma$. Then $\{1/2\} = \alpha_g \in A(g) = \{[0, 1], \alpha_g\}$ with $B(\alpha_g) = [0, 1]$.

We make the following definition with Proposition 4.2 in mind. We say that the map $\overline{A} : (C(I, I), \|\circ\|) \to (K^*, H^*)$ is *dominant lower semicontinuous* at f if for any $\varepsilon > 0$ there exists $\delta > 0$ so that for any g in $C(I, I)$ for which $\|f - g\| < \delta$ and for any α_f in $A(f)$, there exists an α_g in $A(g)$ so that $\alpha_g \subset B_\varepsilon(\alpha_f)$.

Proposition 4.4. *The map $\overline{A} : (C(I, I), \|\circ\|) \to (K^*, H^*)$ given by $f \mapsto \overline{A(f)}$ is dominant lower semicontinuous.*

Proof. Let $f \in C(I, I)$ with $\varepsilon > 0$. Since $\overline{A(f)}$ is compact in (K, H), there exists an $\varepsilon/2$-net of $\overline{A(f)}$ comprised of elements of $A(f)$; let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset A(f)$ be an $\varepsilon/2$-net of $\overline{A(f)}$. By Proposition 4.2, for each $k = 1, 2, 3, \ldots, n$ there exists $\delta_k > 0$ so that, if $\|f - g\| < \delta_k$, then there exists $\alpha_g \in A(g)$ for which $\alpha_g \subset B_{\varepsilon/2}(\alpha_k)$. Set $\delta = \min\{\delta_1, \delta_2, \ldots, \delta_n\}$, and take $\alpha_f \in A(f)$ with $g \in C(I, I)$ such that $\|f - g\| < \delta$. Since $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is an $\varepsilon/2$-net of $\overline{A(f)}$ in (K, H), there exists $1 \leqslant k \leqslant n$ so that $H(\alpha_f, \alpha_k) < \varepsilon/2$. By our choice of δ, there exists $\alpha_g \in A(g)$ so that $\alpha_g \subset B_{\varepsilon/2}(\alpha_k)$. Thus, $\alpha_g \subset B_{\varepsilon/2}(\alpha_k) \subset B_\varepsilon(\alpha_f)$, and our conclusion follows. □

The main result of this section is Theorem 4.5, where we show that, for a residual subset S of functions in $C(I, I)$, we can strengthen Proposition 4.2 by removing the dominance requirement.

Theorem 4.5. *If f is an element of $C(I, I)$ for which one of the following holds, then $\overline{A} : (C(I, I), \|\circ\|) \to (K^*, H^*)$ given by $f \mapsto \overline{A(f)}$ is lower semicontinuous at f:*

(1) $\overline{S(f)} \supseteq P(f)$,
(2) *$\Lambda : (C(I, I), \|\circ\|) \to (K, H)$ given by $f \mapsto \Lambda(f)$ is lower semicontinuous at f.*

We develop Theorem 4.5 through a series of lemmas and propositions. We begin by showing that the set of continuous functions f which satisfies the condition $\overline{S(f)} \supseteq P(f)$ contains a residual subset of $C(I, I)$.

Lemma 4.6. *Let $S = \{f \in C(I, I)\colon\ \overline{S(f)} \supseteq P(f)\}$. Then S contains a residual subset of $C(I, I)$.*

Proof. Let $S_{n,m} = \{f \in C(I, I)\colon\ f^n$ is not unisigned on $B_{1/m}(x)$ for any $x \in P_n(f)\}$. Since $S_{n,m}$ is a dense and open subset of $C(I, I)$ for any natural numbers n and m, it follows that $S^* = \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} S_{n,m} = \{f \in C(I, I)\colon\ S(f) = P(f)\}$ is residual in $C(I, I)$. Since $S^* \subset S$, our conclusion follows. □

Proposition 4.7. *Let $f \in C(I, I)$ with $\alpha \in A(f)$ indecomposable, $\beta = \bigcap_{n\geqslant 0} f^n(\alpha)$ and $\varepsilon > 0$. If $f \in S$, then there exists $\delta > 0$ so that for any g in $C(I, I)$ for which $\|f - g\| < \delta$, there is $\gamma \in A(g)$ such that $H(\beta, \gamma) < \varepsilon$.*

Proof. Let α be an indecomposable asymptotically stable set of $f \in C(I, I)$, and set $\beta = \bigcap_{n\geqslant 0} f^n(\alpha)$ with $\varepsilon > 0$. By Proposition V.21 of [1], β is the disjoint union of connected closed components β_i, for $i = 1, 2, \ldots, d$. Moreover, $f(\beta_i) = \beta_{i+1}$ for $1 \leqslant i \leqslant d - 1$, and $f(\beta_d) = \beta_1$. Now, choose $\beta \subset V \subset B_\varepsilon(\beta)$ open so that $f(\overline{V}) \subset V$, $\beta = \bigcap_{n\geqslant 0} f^n(\overline{V})$ and V is the disjoint union of connected open components V_i, for $i = 1, 2, \ldots, d$, such that $V_i \cap \beta = \beta_i$ for all i, and $f(\overline{V_i}) \subset V_{i+1}$ for $1 \leqslant i \leqslant d-1$ with $f(\overline{V_d}) \subset V_1$. That this decomposition of V is possible follows from Propositions V.15 and V.21 of [1]. Set $\delta = \min\{|x - y|\colon\ x \in f(\overline{V}),\ y \in \delta V\}$. If $g \in C(I, I)$ for which $\|f - g\| < \delta$, then $g(\overline{V}) \subset V$ with $g(\overline{V_i}) \subset V_{i+1}$ for $1 \leqslant i \leqslant d - 1$ and $g(\overline{V_d}) \subset V_1$. Moreover, $F = \bigcap_{n\geqslant 0} g^n(\overline{V})$ is an asymptotically stable set of g, where F is the disjoint union of connected closed components F_i, for $i = 1, 2, \ldots, d$, with $g(F_i) = F_{i+1}$ for $1 \leqslant i \leqslant d - 1$, and $g(F_d) = F_1$. Since $F \in A(g)$, it follows that $F \neq \emptyset$ and $g(F) \subset F$, so that $F_i \neq \emptyset$ for all i. Since $F \subset B_\varepsilon(\beta)$, we have that $F_i \subset B_\varepsilon(\beta_i)$ for all i. Our goal is to show that $\beta_i \subset B_\varepsilon(F_i)$. Since $f^d(\beta_i) = \beta_i$ for all i, there exist x_i and y_i in β_i such that $f^d(x_i) = \min \beta_i = m_i$ and $f^d(y_i) = \max \beta_i = M_i$. We proceed through several cases predicated upon the location of x_i and y_i in β_i.

Case 1. Suppose x_i and y_i can both be chosen in (m_i, M_i). Let C_i be the compact interval with endpoints $\{x_i, y_i\}$. Then $f^d(C_i) = \beta_i \supset C_i$. Set $\delta = \min\{|M_i - \max C_i|, |m_i - \min C_i|\}$. If $g \in C(I, I)$ so that $\|f^d - g^d\| < \min\{\varepsilon, \delta\}$, then $g^d(C_i) \supset C_i$ and $g^d(C_i) \subset F_i$, with $\beta_i \subset B_\varepsilon(g^d(C_i))$. It follows, then, that $\beta_i \subset B_\varepsilon(g^d(C_i)) \subset B_\varepsilon(F_i)$.

Case 2. Suppose $x_i = m_i$ and $y_i < M_i$, or $x_i > m_i$ and $y_i = M_i$. We assume that $x_i = m_i$ and $y_i < M_i$, as the proof of the other possibility is analogous. Since $y_i < M_i$, we have $f^d(y_i) > y_i$, so that g^d sufficiently close to f^d implies $g^d(y_i) > y_i$ and that $|g^d(y_i) - M_i| < \varepsilon$. As $f^d(x_i) = m_i = x_i$, one has that $x_i \in P(f) \subset \overline{S(f)}$, and if g is sufficiently close to f, then $x_i \in B_\varepsilon(P(g^d))$.

This follows from $P(f) \subset \overline{S(f)}$, and Lemma 9 of [9]. Thus, there is $x_i^* \in \Lambda(g^d)$ for which $|x_i^* - x_i| < \varepsilon$. Moreover, since $x_i^* \in \Lambda(g^d)$, we have that $x_i^* \in \bigcap_{n\geqslant 0} g^{nd}(\overline{V_i}) = F_i$. Because F_i is connected, it follows that $g^d([x_i^*, y_i]) \supset [x_i^*, y_i]$. If $C_i = [x_i^*, y_i]$, then $g^d(C_i) \subset F_i$, with $\beta_i \subset B_\varepsilon(g^d(C_i)) \subset B_\varepsilon(F_i)$.

Case 3. Suppose $x_i = m_i$ and $y_i = M_i$, or $x_i = M_i$ and $y_i = m_i$. In either case, we use the condition that $P(f) \subset \overline{S(f)}$ to establish the existence of an interval $[x_i^*, y_i^*] = C_i$ such that $g^d(C_i) \subset F_i$ and $\beta_i \subset B_\varepsilon(g^d(C_i)) \subset B_\varepsilon(F_i)$, as in Case 2.

Case 4. Suppose $y_i = m_i$ and $x_i < M_i$, or $y_i > m_i$ and $x_i = M_i$. We assume that $y_i = m_i$ and $x_i < M_i$, as the proof of the other possibility is analogous. Since $f^d(y_i) = M_i > x_i$, there exists $y_i^* \in B_\varepsilon(y_i)$ so that $f^d(y_i^*) > \max\{M_i - \varepsilon, x_i\}$. Let $\delta_1 > 0$ so that $g^d(y_i^*) > \max\{M_i - \varepsilon, x_i\}$, too, whenever $\|f^d - g^d\| < \delta_1$. Now, $f^d(x_i) = y_i < y_i^*$, so there is $\delta_2 > 0$ for which $g^d(x_i) < y_i^*$ whenever $\|f^d - g^d\| < \delta_2$. If $\delta < \min\{\delta_1, \delta_2\}$ and $C_i = [y_i^*, x_i]$, then $\|f^d - g^d\| < \delta$ implies $g^d(C_i) \supset C_i$, so that $g^d(C_i) \subset F_i$ and $\beta_i \subset B_\varepsilon(g^d(C_i)) \subset B_\varepsilon(F_i)$. $\square$

Proposition 4.8. *Let $f \in C(I, I)$ with $\alpha \in A(f)$ indecomposable and $\varepsilon > 0$. If $f \in S$, then there exists $\delta > 0$ so that for any g in $C(I, I)$ for which $\|f - g\| < \delta$, there is $\gamma \in A(g)$ such that $H(\alpha, \gamma) < \varepsilon$.*

Proof. Take α to be an indecomposable asymptotically stable set of f, and set $\beta = \bigcap_{n\geqslant 0} f^n(\alpha)$, with $\varepsilon > 0$. Let us choose V an open set containing α so that $\alpha \subset V \subset B_\varepsilon(\alpha)$ and $\beta = \bigcap_{n\geqslant 0} f^n(\overline{V})$. Moreover, if β is the disjoint union of connected closed components β_i, for $i = 1, 2, \ldots, d$, such that $f(\beta_i) = \beta_{i+1}$ for $1 \leqslant i \leqslant d - 1$, and $f(\beta_d) = \beta_1$, then V is the disjoint union of connected open components V_i, for $i = 1, 2, \ldots, d$, such that $V_i \cap \beta = \beta_i$ for all i, and $f(\overline{V_i}) \subset V_{i+1}$ for $1 \leqslant i \leqslant d - 1$ and $f(\overline{V_d}) \subset V_1$. Now, with Proposition 4.7 in mind, take $0 < \delta < \min\{|x - y|\colon\ x \in f(\overline{V}),\ \ y \in \delta V\}$ so that $\|f - g\| < \delta$ implies $g(\overline{V}) \subset V$ and $F = \bigcap_{n\geqslant 0} g^n(\overline{V}) \in A(g)$ such that $H(F, \beta) < \varepsilon$. We set $\alpha^* = (\overline{\bigcup_{n\geqslant 0} g^n(\alpha)}) \cup F$, and show that α^* is an asymptotically stable set of g such that $H(\alpha, \alpha^*) < \varepsilon$. That $H(\alpha, \alpha^*) < \varepsilon$ follows from our choice of $\delta > 0$, since $\alpha \subset \alpha^* \subset V \subset B_\varepsilon(\alpha)$. We now show that $g(\alpha^*) \subset \alpha^*$. Let $x \in \alpha^* = (\overline{\bigcup_{n\geqslant 0} g^n(\alpha)}) \cup F$. Since $g(F) \subset F$, let us suppose that $x \in (\overline{\bigcup_{n\geqslant 0} g^n(\alpha)})$. Then there exists $\{x_n\} \subset \bigcup_{n\geqslant 0} g^n(\alpha)$ such that $\lim_{n\to\infty} x_n = x$. Since $x_n \in \bigcup_{n\geqslant 0} g^n(\alpha)$ for any n, it follows that $g(x_n) = y_n \in \bigcup_{n\geqslant 0} g^n(\alpha)$, too. Since g is continuous, $x_n \to x$ implies that $g(x_n) = y_n \to g(x) = y$, so that $y \in \overline{\bigcup_{n\geqslant 0} g^n(\alpha)} \subset \alpha^*$. It now follows from Proposition V.13 of [1] that α^* is an asymptotically stable set of g. $\square$

Proposition 4.9. *Let $f \in C(I, I)$ with $\alpha \in A(f)$ and $\varepsilon > 0$. If $f \in S$, then there exists $\delta > 0$ so that for any g in $C(I, I)$ for which $\|f - g\| < \delta$, there is $\gamma \in A(g)$ such that $H(\alpha, \gamma) < \varepsilon$.*

Proof. From Proposition V.21 of [1], we know that any asymptotically stable set can be expressed as the disjoint union of finitely many indecomposable asymptotically stable sets. This result establishes Proposition 4.9 as an immediate corollary of Proposition 4.8. $\square$

With Proposition 4.9, we are now in a position to readily establish Theorem 4.5.

Proof of Theorem 4.5. Suppose $\overline{S(f)} \supset P(f)$. Since $\overline{A(f)}$ is compact in (K, H), there exists an $\varepsilon/2$-net of $\overline{A(f)}$ comprised of elements of $A(f)$; let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset A(f)$ be an $\varepsilon/2$-net of $\overline{A(f)}$. By Proposition 4.9, for each $k = 1, 2, 3, \ldots, n$ there exists $\delta_k > 0$ so that, if $\|f - g\| < \delta_k$, then there exists $\alpha_g \in A(g)$ for which $H(\alpha_g, \alpha_k) < \varepsilon/2$. Set $\delta = \min\{\delta_1, \delta_2, \ldots, \delta_n\}$, and take $\alpha_f \in A(f)$ with $g \in C(I, I)$ such that $\|f - g\| < \delta$. Since $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is an $\varepsilon/2$-net of $\overline{A(f)}$ in (K, H), there exists $1 \leqslant k \leqslant n$ so that $H(\alpha_f, \alpha_k) < \varepsilon/2$. By our choice of δ, there exists $\alpha_g \in A(g)$ so that $H(\alpha_g, \alpha_k) < \varepsilon/2$. Thus, $H(\alpha_f, \alpha_g) < \varepsilon$, and our conclusion follows.

From [9], we know that $\Lambda : C(I, I) \to K$ is lower semicontinuous at f if and only if $\overline{S(f)} = \Lambda(f)$. Since $\Lambda(f) \supset P(f)$, condition (2) is also sufficient to insure the lower semicontinuity of $\overline{A} : C(I, I) \to K^*$. $\square$

5. Typical behavior of $\overline{A} : C(I, I) \to K^*$

Our main result is the following theorem.

Theorem 5.1. *There exists a residual subset T of $C(I, I)$ so that*

(1) *every function f in T possesses a nontrivial asymptotically stable set,*
(2) *$\overline{A} : C(I, I) \to K^*$ is lower semicontinuous at every element of T, and*
(3) *$\overline{A} : T \to K^*$ is a continuous map.*

We begin by showing that the collection of functions in $C(I, I)$ that do possess a nontrivial asymptotically stable set is both dense and open in $C(I, I)$.

Proposition 5.2. *Let $R = \{f \in C(I, I)\colon\ A(f) - \{I\} \neq \emptyset\}$. Then R is a dense open subset of $C(I, I)$.*

Proof. First, we show that R is open in $C(I, I)$. Let f be any function in R. Since f has an asymptotically stable set α, there exists an open set W containing α such that $f(\overline{W}) \subset W$. Since $f(\overline{W})$ and δW are both compact, there is a positive distance $\varepsilon > 0$ between them. It follows that for any function $g \in C(I, I)$ for which $\|f - g\| < \varepsilon/2$, one has $g(\overline{W}) \subset W$, and that $\alpha_g = \bigcap_{n \geqslant 0} g^n(\overline{W})$ is an asymptotically stable set for g. Thus, g is an element of R, and R is open in $C(I, I)$.

To show that R is dense in $C(I, I)$, let h be any function in $C(I, I)$, with $y \in I$ a fixed point of h and $\varepsilon > 0$. We perturb h on $B_\varepsilon(y)$ to get the new function h^* so that

(1) h^* is a contraction map on $B_{\varepsilon/2}(y)$ with y its necessarily unique fixed point on that set,
(2) $h^*(x) = h(x)$ on $I - B_\varepsilon(y)$, and
(3) h^* is continuous on I with $\|h^* - h\| < \varepsilon$.

It follows that $\{y\}$ is an asymptotically stable set for h^*, so that h^* is an element of R. $\square$

An immediate corollary of Proposition 5.2 is the following result concerning transitive functions.

Corollary 5.3. *Transitive functions are nowhere dense in $C(I, I)$.*

We now prove our main result.

Proof of Theorem 5.1. Let R be the dense open subset of $C(I, I)$ found in Proposition 5.2, with S the residual subset of $C(I, I)$ found in Lemma 4.6. Then $R \cap S \subset C(I, I)$ is residual, with each element of $R \cap S$ satisfying (1) and (2) of Theorem 5.1. From [10], there exists T a residual subset of $R \cap S$ that satisfies (3), too. We must show that T is also residual in $C(I, I)$. Since $R \cap S$ is a residual subset of $C(I, I)$, $R \cap S = C(I, I) - \bigcup_{n=1}^{\infty} F_n$, where F_n is a nowhere dense subset of $C(I, I)$ for each n. Since T is a residual subset of $R \cap S$, $T = \{R \cap S\} - \bigcup_{n=1}^{\infty} T_n$, where T_n is a nowhere dense subset of $R \cap S$ for each n. Since $R \cap S \subset C(I, I)$, each T_n is nowhere dense in $C(I, I)$, too. Thus, $T^c = (\bigcup_{n=1}^{\infty} F_n) \cup (\bigcup_{n=1}^{\infty} T_n)$ is a first category subset of $C(I, I)$, so that $T = C(I, I) - T^c$ is residual there. $\square$

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